Solitons on Compact and Noncompact Spaces in Large Noncommutativity

El Mostapha Sahraoui and El Hassan Saidi

UFR-High Energy Physics, Department of Physics, Faculty of Sciences,
Rabat University, Av. Ibn Battouta B.P. 1014, Morocco.

H-saidi@fsr.ac.ma, sahraoui@fsr.ac.ma

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Abstract

We study solutions at the minima of scalar field potentials for Moyal spaces and torii in the large non-commutativity and interprete these solitons in terms of non-BPS D-branes of string theory. We derive a mass spectrum formula linking different D-branes together on quantum torii and suggest that it describes general systems of D-brane bound states extending the D2-D0 one. Then we propose a shape for the effective potential approaching these quasi-stable bound states. We give the gauge symmetries of these systems of branes and show that they depend on the quantum torii representations.

1 Introduction

It has been conjectured that at the stationary point of the tachyon potential for non-BPS D-branes and for the D-brane-anti-D-brane pair of string theories, the negative energy density cancels the brane tensions [1, 2, 3]. This means that the minimum of the tachyon potential represents the usual vacuum of the closed string theory without any D-branes. This conjecture was studied in [4] using a WZW-like open superstring field theory free of contact term divergences where the condensation of the tachyon field alone seems to give approximatively the vacuum energy. This phenomenon was demonstrated directly in [5] using Witten's string field theory with cubic interaction [6].

Moreover tachyon condensation may also be studied using non-commutative geometry (NC). This latter arises very naturally in string theory when the antisymmetric background field is taken into account [7, 8, 9, 17]. This important result has brought extra connections between the geometry of D-branes and K-homologies on the C^* algebras and has opened new issues in the analysis of non-commutative quantum field theory and string field theory especially in the study of tachyon solitons. Indeed, it has been shown that starting from D-branes of bosonic string theory and turning on an antisymetric NS-NS B-field, we can get a condensation of tachyon fields living on the world volume of the lower dimensional branes. The same is true for non-BPS branes of type II superstrings and for the D–brane-anti-D-brane systems. The main idea of this result is based on: first, Sen's conjecture saying that it is not necessary to know exactly the shape of the tachyon potential, what one really needs is its values at the extrema. Second, the computation of the vaccum energy configurations where the kinetic part of the effective action is neglected in front of the potential term after non-commutative space coordinates rescaling.

On the other side, the scalar field is no more far to know the same analysis. Indeed, it was shown [10] that starting from the non-commutative scalar field action, one cannot only proof the existence of stable solitons but give them an approximate description at large non-commutativity. This is not the end of the story since Harvey *et al* [11, 12] have given, based on the Gopakumar *et al* (GMS) work [10], a more precise recipe for the solitonic solutions in terms of lower dimentional D-branes whose dimensions depend intimately on the manner of turning on the antisymetric NS-NS B-field.

All this material at hands, Gross and Nekrasov [13, 14, 15] have developed the notion of fluxon tubes in string theory by identifying gauge fields as Higgs ones and minimising the energy of BPS D1-D3 system by solving the Bogomoln'y and Nahm equations [18, 19, 20]. Finally one ends with a magnetic fluxon whose tension maches exacltly the fundamental string one.

In [16], Bars *et al* have studied the tachyon condensation in the non-commutative torus and predictd the existence of D2-D0 bound states using Power-Rieffels projectors. We expect that this result is generalised to higher dimensional compact quantum space involving a rich spectrum of bound states. As a first step in this direction we will

consider higher dimentional torii.

The aim of this work is to study NC solitons in higher dimensional compact and noncompact quantum spaces and explore, amongst others, the bound states extending the Bars *et al* solitons [16]. Our analysis will be made in two steps:

1. Build NC solitons in $2l$ dimensional Moyal spaces, for $l \geq 1$. This allows us to, first explore the features of NC solitons in higher quantum spaces and the shape of the effective potential describing these solutions. Second make an idea about the bound states one expects in $2l$ dimensional compact spaces. Finally, this part may be also extended to $(2l + 1)$ quantum spaces where instead of D0-branes, one ends up with electric fluxons; as such this part may be viewed also as an extension of Harvey *et al* analysis of D25-brane[11].

2. Study of tachyon solitons for $2l$ dimensional torii $\mathbb{T}_\theta^{2l}$ and analyse the analogue of D0-D2 bound state considered in [16] for the quantum two torus. Since this later has rational and irrational representations, we consider various realisations of $\mathbb{T}_\theta^{2l}$ and explore the corresponding bound states, their mass spectrum and their underlying gauge symmetries.

The paper is organised as follows: In section 2, we study non-commutative soliton solutions for scalar fields in Moyal plane $\mathbf{R}_\theta^2$ and generalise it to more wide non-commutative spaces. In section 3, we analyse the non-commutative rational and irrational representations of torii in order to give their respective projectors, the key for the tachyon solutions. In section 4, we derive the exact formula of the mass spectrum of the vaccum configurations and give an interpretation for its meaning. The last section is devoted for discussions and conclude with the progress of the present work.

2   Non-commutative solitons on Moyal spaces

In this section, we review the main lines of the NC soliton of a scalar field theory on $\mathbf{R}_\theta^2 \times \mathbf{R}$ and explore how this kind of systems appears in low energy dynamics of string field theory. Then we extend the corresponding results to non commutative scalar field theories on $\mathbf{R}_\theta^{2l} \times \mathbf{R}$ where now $\mathbf{R}_\theta^{2l}$ is a $2l$ dimensional Moyal space whose local coordinates $\left\{ x^I = \left(x^{2i-1}, x^{2i} \right); \quad i = 1, 2, \cdots, l; \quad I = 1, 2, \cdots, 2l \right\}$ are taken such that

$$\left[x^{2i-1}, x^{2i} \right] = \theta_i, \tag{1}$$
$$\left[x^{2i\pm 1}, x^{2j\pm 1} \right] = 0, \tag{2}$$
$$\left[x^{2i}, x^{2j} \right] = 0, \tag{3}$$
$$\left[x^{2i\pm 1}, x^{2j} \right] = 0, \quad i \neq j. \tag{4}$$

In eq (1) the θ_i's are l real numbers which we choose to be positive definite; otherwise one has just to rename the coordinate variables and turning back to the first case. For

example taking θ_1 negative and all the other θ_i's positive, one has just to set $x^1 = x'^2$ and $x^2 = x'^1$ and come back to the initial case where all θ_i's are positive. As a matter of convention notations, we shall use in what follows the convenient normalisation $\left(y^{2i-1} = x^{2i-1}/\sqrt{\theta_i}, \quad y^{2i} = x^{2i}/\sqrt{\theta_i}\right)$; but for commodity we shall continue to denote the y^{2i-1} and y^{2i} variables as x^{2i-1} and x^{2i} respectively. Observe in passing that for non zero θ_i's, $\mathbf{R}_\theta^{2l}$ is a quantum space viewed as an algebra $\mathcal{A}_\theta$ of endomorphisms of the hilbert space $\mathcal{H}$ of harmonic oscillators; i.e $\mathcal{A}_\theta = End(\mathcal{H})$. To tie up this discussion on the θ_i's, note also that they measure the non commutativity of the space variables and have various interpretations. In string theory, the θ_i parameters are roughly speaking linked to the inverse of the NS-NS B-field as

$$\theta^{IJ} = -(2\pi\alpha')^2 \left(\frac{1}{g+2\pi\alpha'B} B \frac{1}{g-2\pi\alpha'B}\right)^{IJ}, \tag{5}$$

$$G_{IJ} = g_{ij} - (2\pi\alpha')^2 \left(Bg^{-1}B\right)_{IJ}, \tag{6}$$

where G_{IJ}, and g_{IJ}, denote the effective open string and the closed string metrics respectively. In the large noncommutativety these equations reduce to

$$\theta^{IJ} = \begin{cases} \left(\frac{1}{B}\right)^{IJ} & i,j = 1, \cdots, 2l \\ 0 & \text{otherwise} \end{cases} \tag{7}$$

$$G_{IJ} = \begin{cases} -(2\pi\alpha')^2 \left(Bg^{-1}B\right)_{IJ} & i,j = 1, \cdots, 2l \\ g_{IJ} & \text{otherwise} \end{cases}. \tag{8}$$

In our present study, the B-field is taken as $B_{IJ} = B_{[I/2]}\Omega_{IJ}$ and similarly $\theta_{IJ} = \theta_{[I/2]}\Omega_{IJ}$, where Ω_{IJ} is the antisymmetric $2l \times 2l$ matrix of the symplectic form.

2.1 Non-commutative soliton on Moyal plane

We start by recalling that the field action $\mathcal{S} = \mathcal{S}(\phi)$ of a scalar field theory on the noncommutative $\mathbf{R}_\theta^2 \times \mathbf{R}$ space-time in the convention notation we are using is:

$$\mathcal{S} = \int_{\mathbf{R}_\theta^2 \times \mathbf{R}} \mathrm{d}^3\mathrm{x} \left(\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + \theta V(*\phi)\right). \tag{9}$$

Here $V(*\phi)$ is the potential of the noncommutative field operator belonging to the noncommutative algebra $\mathcal{A}_\theta$ introduced above. The $*$ product is the usual Moyal product normalised to

$$f(x) * g(x) = \exp\left(\frac{i}{2}\epsilon_{IJ}\partial/\partial x^I \partial/\partial y^J\right) [f(x)g(y)] \mid_{x=y}, \tag{10}$$

where ϵ_{IJ} is the 2×2 antisymmetric matrix. In eq (10), $f(x)$ and $g(x)$ are functions on the Moyal plane which by using the Weyl correspondence may be interpreted as

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matrix operators F and G of the algebra $\mathcal{A} = End(\mathcal{H})$ acting on the Hilbert space $\mathcal{H}$ of the harmonic oscillator. In this correspondence, $f * g$ is replaced by the usual matrix product $F.G$ and the integration with respect to the non-commutative x^I's is translated into the trace in $\mathcal{A}$. The scalar field operator $\phi(x)$ we consider here is interpreted in string theory as the tachyon field $T(x)$ of the open string ending on $D2$ brane. This may be immediatly seen in large non-commutativity limit by comparing eq (9) to the following the tachyon effective action $\mathcal{S} = \mathcal{S}(T(x))$ obtained from string field theory by keeping $T(x)$ and integrating out all other fields

$$\mathcal{S} = \frac{C}{G_S} \int \mathrm{d}^3 \mathrm{x} \sqrt{G} \left(\frac{1}{2} f \left(*T \right) G^{\mu\nu} \partial_\mu T \partial_\nu T + \cdots + \theta V \left(*T \right) \right). \tag{11}$$

In this equation G_S is the open string coupling constant, C is related to the $D2l$-brane mass as $C = G_S M_{\mathrm{D}2l}$ and the effective coupling $f(T)$ is normalised as $f(0) = 0$ and $f(t_{\mathrm{max}}) = 1$ as suggested by Sen's conjecture.

The total energy E of the scalar field theory eq(9) is given by

$$E = \int_{\mathbf{R}^2} \mathrm{d}^2 \mathrm{x} \left[(\partial_i \phi)^2 + \theta V \left(*\phi \right) \right]. \tag{12}$$

In the large θ limit, the kinetic term $(\partial_i \phi)^2$ in eq (12) may be neglected and the stable field configuration is achieved by minimising the scalar potential $V(*\phi)$. Since $V(*\phi)$ is valued in $\mathcal{A}_\theta$, its minimisation is not a simple task as it involves differential Moyal calculus. A tricky soliton solution has been obtained by Gopakumar, Minwalla and Strominger (GMS for short); it is based on taking the scalar field ϕ as [10]

$$\phi(x) = \sum \varphi_n p_n(x), \tag{13}$$

$$V(*\phi) = \sum V(\varphi_n) p_n(x), \tag{14}$$

where the p_i's are mutually orthogonal projectors of $\mathcal{A}_\theta$ and where the φ_i's are the critical values solving $\frac{dV(\varphi)}{d\varphi} = 0$. Using Sen's conjecture for string field theory which suggest that the tachyon potential $V(t)$ has two extrema; one minimum at the origin $t_{\mathrm{min}} = 0$ with $V(t_{\mathrm{min}}) = 0$ and a maximum at t_{max} with $V(t_{\mathrm{max}})$, one looks for special tachyon field configurations of type $T(x) = t_{\mathrm{max}} p(x)$ solving the equation of motion

$$\frac{\mathrm{d}V}{\mathrm{d}t} = 0, \tag{15}$$

which, upon using the identity $p(x) * p(x) = p(x)$, should be understood as

$$\frac{\mathrm{d}V(tp)}{\mathrm{d}T} |_{T=tp} = \left(\frac{\mathrm{d}V(t)}{\mathrm{d}t} \right) p(x). \tag{16}$$

The last identity is a special situation of eq(15) and follows from the fact that any polynomial function F in the projector p, we have

$$F(\lambda p) = F(\lambda) p. \tag{17}$$

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Denoting by $a^{-} \equiv a = \frac{1}{\sqrt{2}} (x^1 - ix^2)$ and $a^{+} = \frac{1}{\sqrt{2}} (x^1 + ix^2)$, a generic level k soliton solution is given by projection operators $p_k^{(\Lambda)}$, defined up to unitary Λ automorphisms, with $\Lambda\Lambda^{+} = \Lambda^{+}\Lambda = I_d$ of $\mathcal{A}$ as follows

$$p_k^{(\Lambda)} \left(a^{\pm} \right) = \Lambda^{+} \left[\sum_{r=0}^{k-1} \frac{1}{r!} \left(a^{+} \right)^r |0\rangle\langle 0| \frac{1}{r!} \left(a \right)^r \right] \Lambda. \tag{18}$$

In this case the soliton solution mimimising (12) is $\phi = \varphi_k \, p_k^{(\Lambda)}$ and the total energy $E_{\min}$ of the configuration is

$$E_{\min} = k\theta V \left(\varphi_k \right). \tag{19}$$

Before going ahead, let us give an interpretation of this result in terms of non-BPS branes and tachyon condensation. For the leading level $k = 1$, eq(18) reduces to $p_1(x) = |0\rangle\langle 0|$ or equivalently by using harmonic oscillator wave functions language

$$p_1 \left(x \right) = 2 \exp \left(-\frac{r^2}{\theta} \right), \tag{20}$$

where we have set $r^2 = x_1^2 + x_2^2$. In this case the tachyon vacuum configuration is

$$T_1(x) = t_{\max} p_1(x). \tag{21}$$

Note that $t_{\max}$ and $t_{\min}$ have an interesting interpretation; they describe respectively an unstable local maximum representing the space filling $D2$-brane, $(V(t_{\max}))$, and a local minimum representing the closed string vacuum without any D-branes $(V(0) = 0)$. This solution extends naturally to the k-th level as shown here below

$$T_k = t_{\max} \left(|0\rangle\langle 0| + |1\rangle\langle 1| + \ldots + |k-1\rangle\langle k-1| \right); \tag{22}$$

it corresponds to k coincident $D0$-branes leading then to a $U(k)$ gauge symmetry.

In what follows we extend the above results for the $D2$-brane on the Moyal plane to the case of a non BPS $D2l$-brane on the $2l$-dimensional Moyal space introduced earlier and look for new solitonic solutions. Later on, we shall also consider non-BPS branes on non-commutative torii and explore their corresponding tachyon solitons.

2.2 General solitons

On Moyal space $\mathbf{R}_\theta^{2l}$, the situation is more general than the previous one and leads to a very rich spectrum containing, in addition to a stable vaccum field configuration, several other quasi-stable configurations. As we will see, these configurations constitute new solitonic solutions not only because they were not considered before but also because they are associated to the $\left(2^l - 2 \right)$ local minima one gets by minimising the total energy

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of D2l-brane in presence of a NS-NS B_{IJ} field of the form $B_{2i-1,2i} = -B_{2i,2i-1} = B_i$ and zero otherwise. Moreover, these configurations turn out to be very close to similar ones we will consider later for toric non-BPS D2l-branes.

The total energy E of the NC scalar field theory in $\mathbf{R}_\theta^{2l} \times \mathbf{R}$ extending eq(12) associated with the Moyal plane, reads up to a global normalisation factor as

$$E = \int_{\mathbf{R}_\theta^{2l}} \mathrm{d}^{2l}\mathrm{x} \left\{ \sum_{j=1}^{l} \prod_{i \neq j} \theta_i \left[(\partial_j \phi)^2 + \theta_j V(*\phi) \right] \right\}. \tag{23}$$

In eq (23) $V(*\phi)$ is as in (12) but the star product is now more general; it is deduced from (10) by substituting ϵ_{IJ} by the $2l \times 2l$ antsymmetric matric $\xi_{2i-1,2i} = 1$; $\quad i = 1, \cdots, l$ and zero otherwise. Moreover, Weyl correspondence implies that the functions f and g are interpreted as operators of the the algebra $\mathcal{A}_{\{\theta_1,\theta_2,...,\theta_l\}} = End(\mathcal{H}^{\otimes l})$ acting on $\mathcal{H}^{\otimes l}$, the tensorial Hilbert space of l harmonic oscillators. As these l oscillators are uncoupled, the algebra $\mathcal{A}_{\{\theta_1,\theta_2,...,\theta_l\}} = End(\mathcal{H}^{\otimes l})$ split as a product of the algebra factors $\mathcal{A}_{\theta_i}$; that is $\mathcal{A}_{\{\theta_1,\theta_2,...,\theta_l\}} = \otimes_{i \geq 1} \mathcal{A}_{\theta_i}$. Using this feature, we can rewrite eq (23) in a remarkable form by introducing the following E_j energies

$$E_j = \int_{\mathbf{R}_\theta^{2l}} \mathrm{d}^{2l}\mathrm{x} \left\{ (\partial_j \phi)^2 + \theta_j V(*\phi) \right\}, \tag{24}$$

in terms of which, the total energy E reads then as

$$E = \sum_{j=1}^{l} \mu_j E_j, \tag{25}$$

where μ_j is given by the product of all θ_i's divided by θ_j; i.e.

$$\mu_j = \prod_{i \neq j} \theta_i = \frac{\left(\prod_{i=1}^{l} \theta \right)}{\theta_j} \equiv \frac{\Theta^l}{\theta_j}. \tag{26}$$

In the case where all the θ_i's are positive definite, which is our hypothesis here, the minimum of the total energy E in eq(25) is achieved by taking the minima of all E_i's. These are easily obtained since for a given fixed j, E_j is quite similar to the energy of the NC soliton we have studied for the case of a scalar field theory on $\mathbf{R}_\theta^2 \times \mathbf{R}$. Thus taking the large θ_j limits for all j values, the kinetic energy $(\partial_j \phi)^2$ of eqs (23,24) can be neglegted and the stable configuration is then given by minimising $V(*\phi)$. Extending the GMS formalism to our case where we have l harmonic oscillators, the soliton solution take the following form

$$\phi(x_1, \cdots, x_{2l}) = \sum_{n_1, \cdots, n_1 \geq 0} \varphi_{(n_1, \cdots, n_l)} P_{(n_1, \cdots, n_l)}(x_1, \cdots, x_{2l}); \tag{27}$$

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where now the $P_{(n_1,\cdots,n_l)}$'s are mutually orthogonal projectors of $\mathcal{A}_{\{\theta_1,\theta_2,\ldots,\theta_l\}}$ and where the $\varphi_{(n_1,\cdots,n_l)}$'s are as in the Moyal plane analysis. Moreover, introducing the annihilation $a_i^- = a_i = \sqrt{\frac{1}{2}}(x^{2i-1} - ix^{2i})$ and creation $a_i^+ = \sqrt{\frac{1}{2}}(x^{2i-1} + ix^{2i})$ operators, one may here also write the $P_{(n_1,\cdots,n_l)}$'s in a form similar to eq(18). At a given multi-level $\mathbf{k} = (k_1,\cdots,k_l)$, P_k is defined up to an automorphism Λ of $\mathcal{A}_{\{\theta_1,\theta_2,\ldots,\theta_l\}}$, and reads as

$$
P_k\left(a_1^\pm,\cdots,a_l^\pm\right) = \Lambda^+\left(\sum_{r_1=0}^{k_1-1}\sum_{r_2=0}^{k_2-1}\cdots\sum_{r_l=0}^{k_l-1}\right.
$$
$$
\left.\left\{\prod_i^l\left[\frac{(a_i^+)^{r_i}}{(r_i)!}\right]|0,\cdots,0\rangle\langle0,\cdots,0|\prod_j^l\left[\frac{(a_i)^{r_i}}{(r_i)!}\right]\right\}\right)\Lambda. \quad (28)
$$

Note that as the l harmonic oscillators are uncoupled, we can simplify the above relation by using the factorisation property $\mathcal{A}_{\{\theta_1,\theta_2,\ldots,\theta_l\}} = \overset{l}{\underset{i=1}{\otimes}}\mathcal{A}_{\theta_i}$; that is

$$
P_k^{(\Lambda)} = \overset{l}{\underset{i=1}{\otimes}}\left(P_{ki}^{(\Lambda i)}\right); \quad (29)
$$

where $P_{ki}^{(\Lambda i)}$ are projectors in the i-th Hilbert space defined up to Λ_i automorphisms of $\mathcal{A}_{\theta_i}$ and read as

$$
P_{ki}^{(\Lambda i)} = \Lambda_i^+\left(\sum_{r_i}^{k_i-1}\frac{\left(a_i^+\right)^{r_i}}{(r_i)!}|0\rangle\langle0|\frac{(a_i)^{r_i}}{(r_i)!}\right)\Lambda_i. \quad (30)
$$

In terms of these projectors, the configuration minimising eq (23) is then

$$
\phi\left(a_1^\pm,\cdots,a_l^\pm\right) = \varphi_{\mathbf{k}}\left[P_{k_1}^{(\Lambda_1)}\left(a_1^\pm\right)\otimes P_{k_2}^{(\Lambda_2)}\left(a_2^\pm\right)\otimes\cdots\otimes P_{k_l}^{(\Lambda_l)}\left(a_l^\pm\right)\right], \quad (31)
$$

and the total energy $E^{(0)}$ of the stable solitonic field configuration to which we shall also refer to as $E_{\min}^{(0)}$ is

$$
E_{\min}^{(0)} = \prod_{i=1}^l(k_i\theta_i)V(\varphi_{\mathbf{k}}) = k\Theta^l V(\varphi_{\mathbf{k}}), \quad (32)
$$

where we have also set $k = \prod_{i=1}^l k_i$. Up to now this analysis seems to be a generalisation of the analysis performed for the Moyal plane. However this is not the full story since the minimisation of the total energy eq (23) leads also to other local minima depending on the various ways large non-commutativity is taken. In what follows, we study these local minima and explore the field configurations associated with them as well as their interpretation in terms of non-BPS D-branes.

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To start reconsider eq (23) and reexamine all its possible local minima. Since all the coefficients μ_j in front of E_j are positive, then the total energy minimum $E_{\min}$ obtained by taking the minimum of all E_j's. In other words $E_{\min}$, which in addition to the kinetic and potential energies depends moreover on the magnitudes of the θ_i's parametes, is given by

$$E_{\min}(\theta_1, \theta_2, ..., \theta_l) = \sum_{j=1}^{l} \mu_j (E_j)_{\min}. \tag{33}$$

From this equation, one clearly see that the value of $E_{\min}$ depends on the ways the large non-commutativity limit is taken. If one adopts a strong definition of large non-commutativity by requiring all θ_i's large, then all $(E_j)_{\min}$'s are equal to $k_j \theta_j V(\varphi_k)$ and so one discovers the absolute minimum $E_{\min}^{(0)}$ given by eq(32). However, if one adopts a weaker definition for large non-commutativity by requiring that at least one of the θ_i's is large, then the above equation will have several local minima. Let us determine with explicit details the two leading ones and give the general result using iteration techniques.

(1) $E_{\min}^{(1)}$ local minima energy

This energy is obtained from eq(33) by taking all θ_i's large except one of them, say θ_j for some given j, which taken finite. Since j can take l values, the $E_{\min}^{(1)}$ energy is l-th degenerate. Indeed, within this limit one has $(E_i)_{\min, i \neq j} = k_i \theta_i V(\varphi_k)$ in agreement with eqs (23,24) and can usually set the $(E_j)_{\min}$ energy associated to the finite θ_j as given by a derivation above $k_j \theta_j V(\varphi_k)$. Setting $k_n \theta_n V(\varphi_k) = E_n^0$ and $(E_j)_{\min} = E_j^0 + \delta E_j^0$, where δE_j^0 is the gap energy, then putting back into eq(33), one gets

$$(E_{\min}^{(1)})_j = E_{\min}^{(0)} + \delta E_j^0; \quad j = 1, ..., l. \tag{34}$$

Degeneracy of $(E_{\min}^{(1)})_j$ is ensured if one assumes that all δE_j^0 are equal otherwise the degeneracy is rised either partially or totaly depending on the values of δE_j^0 and so one ends with different quasi-stable field configurations.

(2) $E_{\min}^{(2)}$ local minima energy

In tis case the energy $E_{\min}^{(2)}$ is obtained from eq(33) by taking all θ_i's large except two of them, say θ_m and θ_n which are taken to be finite. Similar analysis as before shows that $E_{\min}^{(2)}$ depends on two indices m and n and reads as

$$(E_{\min}^{(2)})_{\{m,n\}} = E_{\min}^{(0)} + \delta E_m^0 + \delta E_n^0; \quad m, n = 1, ..., l. \tag{35}$$

If we assume that δE_m^0 and δE_n^0 are equal, then the corresponding configuration is $\frac{l(l-1)}{2}$ degenerate.

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More generally if we assume that large non-commutativity is achieved by taking the set of parameters $\{\theta_{i_1}, \theta_{i_2}, ..., \theta_{i_s}\}$ finite and $\{\theta_{i_{s+1}}, \theta_{i_{s+2}}, ..., \theta_{i_l}\}$ large. The energy of the local minimum is $(E_{\min}^{(s)})_{\{i_1, i_2, ..., i_s\}}$ reads as

$$(E_{\min}^{(s)})_{\{i_1, i_2, ..., i_s\}} = E_{\min}^{(0)} + \sum_{1 \leq j \leq s} \delta E_{i_j}^0 \quad ; i_j = 1, ..., l; 0 \leq s \leq l - 1. \qquad (36)$$

(3) $E_{\min}^{(s)}$ **local minima energy;** $3 \leq s \leq l - 1$.

Here we give a recaputilating table (a) where we put the energies of local minima, their maximal degeneracies as well as a potential curve $V(\phi)$ representing these minima

Minima energies	# of degenerate states
$E_{\min}^{(0)}$	1
$E_{\min}^{(1)}$	l
$E_{\min}^{(2)}, ...$	$\frac{l(l-1)}{2}, ...$
$E_{\min}^{(s)}$	$\frac{l!}{(l-s)!s!}$
$E_{\min}^{(l-1)}$	l

Table (a)

The total number of local minima $(E_{\min}^{(s)})_{\{i_1, i_2, ..., i_s\}}$, $(1 \leq s \leq l - 1)$, is then $(2^l - 2)$ to which one should add the absolute minimum $E_{\min}^{(0)}$ and the upper bound where all θ_i's are taken finite. On the figure 1 representing a potential curve, which may be though of as the potential of the non-commutative scalar field in Moyal space, we have represented the various local local minima. In string field theory these local minima might be interpreted as associated with non-BPS states; the recipe is that to the absolute minimum we encounter $\prod_{i=1}^{l}(k_i)$ non-BPS D0-brane with a $U(\prod_{i=1}^{l}(k_i))$ gauge symmetry while for the leading local minima of energies $(E_{\min}^{(1)})_j$, we have, for fixed j, $\prod_{i=1, i \neq j}^{l}(k_i) = \prod_{i=1}^{l}(k_i)/k_j$ non-BPS D2-branes with a $U(\prod_{i=1}^{l}(k_i)/k_j)$ gauge symmetry and whose world volume includes the (x^{2j-1}, x^{2j}) Moyal plane itself contained in the $2l$-dimensional Moyal space. Therefore, the total gauge group of non-BPS D2-branes is the cross product of $U(\prod_{i=1}^{l}(k_i)/k_j)$, that is $\otimes_{j=1}^{l} U(\prod_{i=1}^{l}(k_i)/k_j)$. More generally to the local minima of energies $(E_{\min}^{(s)})_{\{i_1, i_2, ..., i_s\}}$, $(0 \leq s \leq l)$, we have for a given configuration $(\prod_{i=1}^{l} k_i)/(\prod_{n=1}^{s} k_{i_n})$ non-BPS D2s-branes with a $U\left((\prod_{i=1}^{l} k_i)/(\prod_{n=1}^{s} k_{i_n})\right)$ gauge symmetry. The total gauge group G_s may here also be written down; it is given by the tensor product of the $\frac{l!}{(l-s)!s!}$ possible factors. It reads as

$$G_s = \otimes_{j=1}^{s} \otimes_{i_j=1}^{l} U\left(\frac{\prod_{i=1}^{l}(k_i)}{\prod_{n=1}^{s} k_{i_n}}\right). \qquad (37)$$
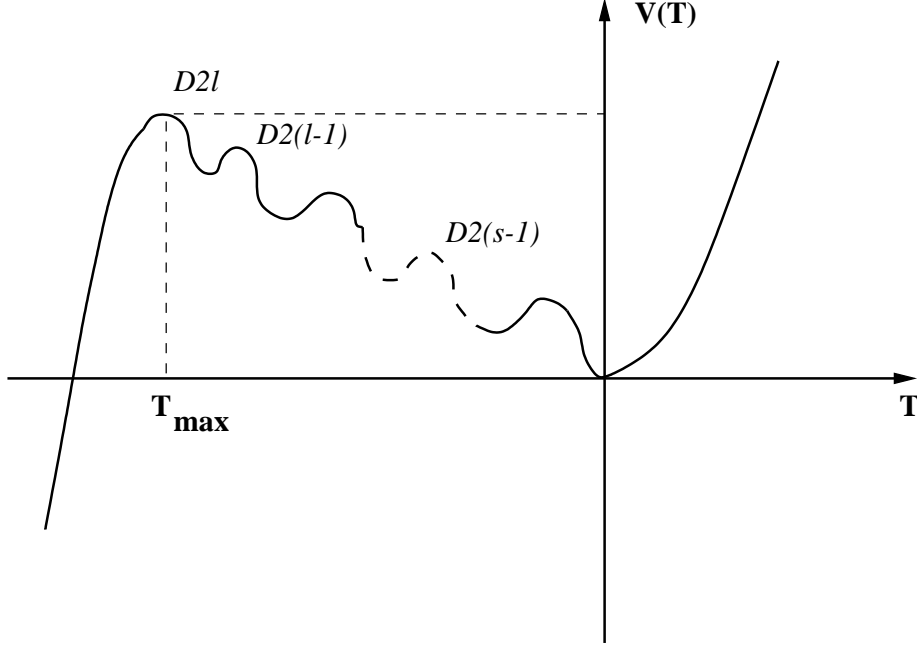
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Figure 1: the suggested shape of the potentil tachyon describing the decay of D2l-brane. To each local maximum s is associated a non-BPS D2s-brane.

Now we turn to give the corresponding field configurations associated to these local minima. These solutions may be written down explicitly by using the appropriate projector opertors on these vacuua. To see how these are built let us first study the leading examples, then extends the results to all local minima. For the absolute minimum $E_{\min}^{(0)} = \left(\prod_{i=1}^{l} k_i \right) \Theta^l V \left(\varphi \right)$, the corresponding soliton denoted as $\phi^{(0)} = \phi_j^{(0)}(\{a_i^{\pm}\})$, was already calculated as shown in eq(31); it reads as

$$\phi^{(0)} = \varphi P_{k_1} \otimes P_{k_2} \otimes \ldots \otimes P_{k_l}, \tag{38}$$

where we have dropped the Λ_i indices on the P_{k_i}'s for simplicity. This is also equivalent to choose Λ_i's as the identity operator This configuration is a stable non-degenerate state and may be thought of as the closed string vacuum in tachyon condensation framework. For the field configurations associated to the local energy minima $E_j^{(1)}$ eq(34), one has l quasi-stable states which we denote as $\phi_j^{(1)} = \phi_j^{(1)}(\{a_i^{\pm}\}_{i \neq j})$. They are

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are given by

$$\phi_1^{(1)} \quad \smile \quad P_1^{(1)} = \mathbf{1} \otimes P_{k_2} \otimes \ldots \otimes P_{k_l} \tag{39}$$

$$\vdots$$

$$\phi_n^{(1)} \quad \smile \quad P_n^{(1)} = P_{k_1} \otimes P_{k_2} \otimes \ldots \otimes \mathbf{1}_n \otimes \ldots \otimes P_{k_l} \tag{40}$$

$$\vdots$$

$$\phi_l^{(1)} \quad \smile \quad P_l^{(1)} = P_{k_1} \otimes P_{k_2} \otimes \ldots \otimes P_{k_{l-1}} \otimes \mathbf{1}, \tag{41}$$

where $P_j^{(1)}$ stands for the projector operator the first local minimum with an identity operator at the j-th position. Similarly, we have $l(l-1)/2$ quasi-stable states $\phi_{\{j_1, j_2\}}^{(2)} = \phi_{\{j_1, j_2\}}^{(2)}(\{a_i^{\pm}\}_{i \neq \{j_1, j_2\}})$ associated with the local minima energy $E_{\min}^{(2)}$; they read as

$$\phi_{\{1,2\}}^{(2)} \quad \smile \quad P_{\{1,2\}}^{(2)} = \mathbf{1} \otimes \mathbf{1} \otimes P_{k_3} \otimes \ldots \otimes P_{k_l} \tag{42}$$

$$\vdots$$

$$\phi_{\{j_1, j_2\}}^{(2)} \quad \smile \quad P_{\{j_1, j_2\}}^{(2)} = P_{k_1} \otimes P_{k_2} \otimes \ldots \otimes \mathbf{1}_{j_1} \otimes \ldots \otimes \mathbf{1}_{j_2} \otimes \ldots \otimes P_{k_l} \tag{43}$$

$$\vdots$$

$$\phi_{\{l-1_1, l\}}^{(2)} \quad \smile \quad P_{\{l-1_1, l\}}^{(2)} = P_{k_1} \otimes P_{k_2} \otimes \ldots \otimes P_{k_{l-1}} \otimes \mathbf{1}_{l-1} \otimes \mathbf{1}_l. \tag{44}$$

More generally, the vacuum field configurations $\phi_{\{j_1, j_2, \ldots, j_s\}}^{(s)} = \phi_{\{j_1, j_2, \ldots, j_s\}}^{(s)}\{a_i^{\pm}\})$, for $i \neq \{j_1, j_2, \ldots, j_s\}$ and $0 \leq s \leq l-1$, associated with a generic local minima of energy $(E_{\min}^{(s)})_{\{j_1, j_2, \ldots, j_s\}}$ are given by

$$\phi_{\{j_1, j_2, \ldots, j_s\}}^{(s)} \smile P_{\{j_1, j_2, \ldots, j_s\}}^{(s)} = P_{k_1} \otimes \ldots \otimes \mathbf{1}_{j_1} \otimes \ldots \otimes P_{k_n} \otimes \ldots \otimes \mathbf{1}_{j_s} \otimes \ldots \otimes P_{k_l}. \tag{45}$$

The soliton solutions we have desribed possede an interesting application in string field theory [11] where the vacuum scalar field configurations are identified with tachyon solitons of various non-BPS D-branes.

To conclude this section we should say that all the analysis concerning the NC soliton of D2-brane on the Moyal plane and which was interpreted as describing D0-branes which appear after tachyon condensation giving the correct D0-brane mass as well as their number in the vacuum state extends naturally to the case of D2l-branes on Moyal spaces. The novelty here is that one has in addition to the D0-branes and D2l ones other quasi-stable configurations which we have interpreted as extra D2s-branes; $1 \leq s \leq l-1$, living altogether with D0 and D2l ones. Moreover given a D2s-brane, one distinguishes different world volumes for these branes; l kinds of D2-branes, $l(l-1)/2$ kinds of D4-branes and so on. Decomposing the $P_{\{j_1, j_2, \ldots, j_s\}}^{(s)}$ projectors into mutually irreducible orthogonal $\chi_{\{r_1, r_2, \ldots, r_s\}}^{(s)}$ ones as herebelow

$$P_{\{j_1, j_2, \ldots, j_s\}}^{(s)} = \sum_{r_1=0}^{j_1-1} \sum_{r_2=0}^{j_2-1} \ldots \sum_{r_l=0}^{j_l-1} \chi_{\{r_1, r_2, \ldots, r_s\}}^{(s)}, \tag{46}$$

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with

$$(\chi^{(s)}_{\{r_1,r_2,\ldots,r_s\}})^2 = \chi^{(s)}_{\{r_1,r_2,\ldots,r_s\}}; \quad (\chi^{(s)}_{\{r_1,r_2,\ldots,r_s\}})^\dagger = \chi^{(s)}_{\{r_1,r_2,\ldots,r_s\}} \tag{47}$$

$$\chi^{(s_1)}_{\{r_1,r_2,\ldots,r_{s_1}\}}\chi^{(s_2)}_{\{q_1,q_2,\ldots,q_{s_2}\}} = 0; \qquad \{^{s_1\neq s_2 \text{ or}}_{s_1=s_2 \text{ with } \{r_1,r_2,\ldots,r_{s_1}\}\neq\{q_1,q_2,\ldots,q_{s_2}\}} \tag{48}$$

or equivalently by rewriting them in a short form as

$$P^{(s)}_{\mathbf{j}} = \sum_{\mathbf{r}}^{\mathbf{j-1}} \chi^{(s)}_{\mathbf{r}}, \tag{49}$$

and

$$(\chi^{(s)}_{\mathbf{r}})^2 = \chi^{(s)}_{\mathbf{r}}; \quad (\chi^{(s)}_{\mathbf{r}})^\dagger = \chi^{(s)}_{\mathbf{r}}$$
$$\chi^{(s_1)}_{\mathbf{r_1}}\chi^{(s_2)}_{\mathbf{r_2}} = 0; \quad for \ s_1 \neq s_2 \text{ or } s_1 = s_2 \text{ but } \mathbf{r_1} \neq \mathbf{r_2}, \tag{50}$$

then the open string wave function Ψ can be projected into pieces $\chi^{(s_1)}_{\mathbf{r}}\Psi\chi^{(s_2)}_{\mathbf{q}}$ representing the open strings interpolating from a non-BPS D2s_1-brane with data $\mathbf{r} = (r_1, r_2, \ldots, r_{s_1})$ to the non BPS D2s_2 one with data $\mathbf{q} = (q_1, q_2, \ldots, q_{s_1})$. Therefore one has a variety of open strings ending on D-branes; in particular D0-D0, D0-D2, D0-D4,$\cdots$, D2-D2, D2-D4,$\cdots$; D4-D4,$\cdots$ and so on.

3 Solitons in non-commutative torus $\mathbb{T}^{2l}_\theta$

Here we want to extend the analysis we have made for Moyal space to the non-commutative torus $\mathbb{T}^{2l}_\theta$. Like for eqs (1), the $\mathbb{T}^{2l}_\theta$ we will be considering is roughly speaking given by the product of l non-commutative two dimensional torii $\mathbb{T}^2_{\theta_i}$. In other words the non-commutative $\mathbb{T}^{2l}_\theta$ is generated by a system of l unitary pairs (U_i, V_i) satisfying the algebra

$$U_n V_n = \mathrm{e}^{-i2\pi\theta_n} V_n U_n \tag{51}$$
$$U_n V_m = V_m U_n; \quad n \neq m, \tag{52}$$

for which we will shall refer from now on as $\mathfrak{A}_\theta$. Note that because of eq(52), the non-commutative algebra $\mathfrak{A}_\theta$ may be also defined as the tensor product of l factors $\mathfrak{A}_{\theta_i}$; $\mathfrak{A}_\theta = \otimes_{i=1}^l \mathfrak{A}_{\theta_i}$; Each $\mathfrak{A}_{\theta_i}$ factor is associated with the non-commutative torus $\mathbb{T}^2_{\theta_i}$; and the corresponding (U_i, V_i) pairs are realised as the exponentials of the non-commutative coordinates (x^{2i-1}, x^{2i}) of $\mathbb{T}^2_{\theta_i}$. For later use we prefer to denote the coordinates of the non-commutative torus by the capital letters (X^{2i-1}, X^{2i}) while those of the commutative ones by small letters. Thus we have

$$U_i = \mathrm{e}^{\frac{i2\pi}{R_{2i-1}}X^{2i-1}}; \quad V_i = \mathrm{e}^{\frac{i2\pi}{R_{2i}}X^{2i}}, \quad i = 1, \ldots, l \tag{53}$$

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where the R_j's are the one cycles radii of the $2l$-dimensional torus. Note also that U_i and V_i generators have different representations according to whether the θ_i's are rational or irrational. In what follows we shall use both of these representations; this is why we first review them briefly on the simple case of $\mathbb{T}^2_{\theta_i}$ torus.

Rational representations

This kind of representations corresponds to rational values of $\theta_i = q_i/p_i$, where p_i and q_i are mutually coprime integers. The U_i and V_i generators are given by the following finite $p_i \times p_i$ matrices

$$U_i = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & \omega_i & 0 & \cdots & 0 \\ 0 & 0 & \omega_i^2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & 0 & 0 & \cdots & \omega_i^{p_i-1} \end{bmatrix} ; \quad V_i = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{bmatrix} \quad (54)$$

where $\omega_i = e^{i2\pi q_i/p_i}$. Note in passing that $U_i^{p_i}$ and $V_i^{p_i}$ act as the $p_i \times p_i$ identity operator I and so any element a_i of the non-commutative algebra $\mathfrak{A}_{\theta_i}$ associated to $\mathbb{T}^2_{\theta_i}$ has a finite expansion

$$a_i = \sum_{n,m=0}^{p-1} (a_i)_{nm} U_i^n V_i^m. \quad (55)$$

Note that in the matrix representation presented above, the U_i generator is given by a diagonal matrix; a feature which allows to build the usual rank k_i projector $(\Pi_i)_{k_i} = \text{diag}(1, 1, \ldots, 1, 0, \ldots, 0)$ as a series of the U_i's, i.e.

$$(\Pi_i)_{k_i} = \sum_{n=0}^{p} (a_i)_{n0} U_i^n. \quad (56)$$

A direct check shows that the $(a_i)_{n0}$ coefficients are given by $(a_i)_{n0} = \frac{1}{p_i} \frac{1 - \omega_i^{-nk}}{1 - \omega_i^{-n}}$. Note moreover that the trace on $\mathfrak{A}_{\theta_i}$ is defined as $\text{Tr}(\Pi_i)_{k_i} = (a_i)_{00} = k_i$. Thus the range of k_i is $0 \leq k_i \leq p_i$ and is interpreted as the number of D0-branes one obtains from the study of a D2-brane on the non-commutative $\mathbb{T}^2_{\theta_i}$.

Irrational representations

The generalisation of the previous case to irrational θ_i's is not automatic and turns out to have interesting interpretations in terms of branes bound states. Following the same lines as for the rational case by working in a representation in which U_i is diagonal and V_i is not, one has the following

$$\begin{aligned} \langle x'_{2i-1}|U_i|x_{2i-1}\rangle &= e^{i2\pi x_1}\delta\left(x'_{2i-1} - x_{2i-1}\right); \\ \langle x'_{2i-1}|V_i|x_{2i-1}\rangle &= \delta\left(x_{2i-1} + \theta_i - x'_{2i-1}\right), \end{aligned} \quad (57)$$

where we have set $R_I = 1$ for commodity. Note in passing that here also U_i is a diagonal operator while V_i is not; they depend on the x^1 variable only. To construct the projector operators on the position space generated by the continuous basis vectors $\{|x_{2i-1}\rangle \times |x_{2i}\rangle\}$, one may consider in a first attempt functions of the diagonal operator U_i. A choice of the function $f(U_i)$ is given by

$$
\begin{aligned}
\langle x'_{2i-1}|(\Pi_i)|x_{2i-1}\rangle &= \langle x'_{2i-1}|f(U_i)|x_{2i-1}\rangle \\
&= \theta^{IJ} = \begin{cases} \kappa_i \delta\left(x'_{2i-1} - x_{2i-1}\right), & 0 \leq x_{2i-1} \leq \kappa_i \\ 0 & \kappa_i < x_{2i-1} \leq 1 \end{cases}
\end{aligned} \tag{58}
$$

κ_i is a priori a real parameter lying between zero and one. This choice of (Π_i) ensures that it is hermitian, $(\Pi_i)^2 = (\Pi_i)$ but still fails as,in general, the trace $\mathrm{Tr}(\Pi_i)$ is not an integer

$$
\mathrm{Tr}(\Pi_i) = \int \mathrm{d}x_{2i-1} \langle x_{2i-1}|(\Pi_i)|x_{2i-1}\rangle = \kappa_i. \tag{59}
$$

This trace is not acceptable, it contradicts the expected spectrum dictated by the group $\mathbf{K}_0\left(\mathfrak{A}_{\theta_i}\right) = \mathbb{Z} + \theta_i \mathbb{Z}$, since κ_i is not quantised. To overcome this difficulty one should use both the U_i and V_i operators instead of using U_i alone; this will allow to also incorporate the non-commutativity in the game. A class of solutions for the projector operators in agreement with $\mathbf{K}_0\left(\mathfrak{A}_{\theta_i}\right)$ has been constructed in [16]. It extends the Powers-Rieffel projectors and reads as

$$
\mathcal{P}_{\{n_i + m_i\theta_i\}} = (V_i^{m_i})^+ \left(g\left(U_i\right)\right)^+ + f\left(U_i\right) + g\left(U_i\right)V_i^{m_i}; \tag{60}
$$

where the function $f(U_i)$ and $g(U_i)$ are given by

$$
f\left(U_i\right) = \begin{cases} x^{2i-1}/\epsilon_i & x^{2i-1} \in [0, \epsilon_i] \\ 1 & x^{2i-1} \in [\epsilon_i, \theta_i] \\ 1 - \left(x^{2i-1} - (n_i + m_i\theta_i)\right)/\epsilon_i & x^{2i-1} \in [\theta_i, \theta_i + \epsilon_i] \\ 0 & x^{2i-1} \in [\theta_i + \epsilon_i, 1] \end{cases} \tag{61}
$$

$$
g\left(U_i\right) = \begin{cases} \sqrt{f\left(U_i\right)\left(1 - f\left(U_i\right)\right)} & x^{2i-1} \in [0, \epsilon_i] \\ 0 & x^{2i-1} \in [\epsilon_i, 1] \end{cases}. \tag{62}
$$

In these eqs ϵ_i is a small parameter which physically may be interpreted as a regulation parameter. Having given the representations of $\mathfrak{A}_{\theta_i}$ for $\mathbb{T}^2_{\theta_i}$, we turn now to extend them to $\mathbb{T}^{2l}_\theta$. For fixed l, we have generally 2^l possibilities depending on whether the θ_i's are rational or irrational. If all θ_i's are rational, i.e. $\theta_i = q_i/p_i$ the U_i and V_i are given by similar eqs to eq (54). If instead all θ_i's are irrational, the U_i's and V_i's are given by

$$
\begin{aligned}
\langle \mathbf{x}'|U_i|\mathbf{x}\rangle &= \mathrm{e}^{i2\pi x_i}\delta\left(\mathbf{x}' - \mathbf{x}\right), \tag{63} \\
\langle \mathbf{x}'|V_i|\mathbf{x}\rangle &= \delta^{2l}\left(\mathbf{x} + \theta_i - \mathbf{x}'\right). \tag{64}
\end{aligned}
$$

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We can also have the case where part of the θ_i's are rational and the others are irrational. In this case the U_i' s and V_i's are given by mixing the representations (54) and (57).

The projectors $\mathcal{P}_{\{\theta_1,...,\theta_l\}}$ on the space position basis $\{|\mathbf{x}\rangle = |(x_1, x_2, ..., x_{2l-1}, x_{2l})\rangle\}$ for $\mathbb{T}_\theta^{2l}$ have then several forms depending on whether the θ_i's are rational or irrational. Denoting by $\mathcal{P}_{\{\theta_i\}}$ the projector operator associated to θ_i which is given by either eq (56) or (60), we have

$$\mathcal{P}_{\{\theta_1,...,\theta_l\}} = \overset{l}{\underset{i=1}{\otimes}} \mathcal{P}_{\{\theta_i\}}. \tag{65}$$

From eq (65), one learns that there are a priori 2^l solutions. However if one identifies operators that are related under permutations of positions, one ends then with l differents objects. Note that the trace of eq (65) is given by the trace on the individual projectors $\mathcal{P}_{\{\theta_i\}}$; i.e

$$\mathrm{Tr}\mathcal{P}_{\{\theta_1,...,\theta_l\}} = \prod_{i=1}^{l} \mathrm{Tr}\mathcal{P}_{\{\theta_i\}}. \tag{66}$$

4 Non-BPS Toric D$2l-$Branes

Consider a non-BPS D$2l$-brane on the $2l$ dimensional non-commutative torus $\mathbb{T}_\theta^{2l}$ given by eq(53) and study the field configurations minimising the total energy $E(T)$ of the tachyon living on the world volume of the brane. Starting from the D$2l$-brane of string theory ($0 \leq l \leq 12$ for the bosonoic case and $0 \leq l \leq 4$ for type IIB) in presence of $B_{\mu\nu}$ field chosen as in the case of the Moyal space, the string field theory effective action $\mathcal{S} = \mathcal{S}(T(x))$, keeping only the tachyon field $T(x)$ and integrating out all other fields, reads as

$$\mathcal{S} = \frac{C_{\mathrm{D}2l}}{G_S} \int \mathrm{d}^{2l+1}\mathrm{x}\sqrt{G} \left(\frac{1}{2} f\left(*T\right) G^{\mu\nu} \partial_\mu T \partial_\nu T + \cdots + V\left(*T\right) \right), \tag{67}$$

where $G_{\mu\nu}$, G_S, $C_{\mathrm{D}2l}$ and the factor $f(t)$ are as in eq(11).

In large non-commutativity, the kinetic term of the tachyon is neglected and so the total energy $E(T)$ reduces to

$$E(T) = M_{\mathrm{D}2l}\mathrm{Tr}\ V(T), \tag{68}$$

where $M_{\mathrm{D}2l}$ denotes the mass of the original D$2l$-brane and the trace is normalysed as $\mathrm{Tr}\ \mathbf{1} = 1$. Extremisation of $E(T)$ is achieved as usual by using the GMS approach which shows that tachyon field configuration are proportional to projectors in the $\mathfrak{A}_\theta$ non-commutative algebra. The interpretation of the solution field configurations are given by Sen's conjecture, where the original D$2l$-brane is interpreted as to correspond

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to $T = t_{\max}\mathbf{1}$ and the complete tachyon condensation $(T = 0)$ to the decay at the vacuum.

In the case of non-BPS D$2l$-brane on the $2l$ dimensional non-commutative torus $\mathbb{T}_\theta^{2l}$ we are interested in here, the tachyon field configurations extremising eq (67) is given by $T = t_{\max}\mathcal{P}_{\{\theta_1,\dots,\theta_l\}}$. Using eq (66) and taking all θ_i's irrational, the total energy of the soliton is

$$E_{\text{total}} \equiv E\left(\mathcal{P}_{\{n_1+m_1\theta_1,\dots,n_l+m_l\theta_l\}}\right) = M_{\text{D}2l} \prod_{i=1}^{l} (n_i + m_i\theta_i). \tag{69}$$

Taking into account the fact that $0 \leq n_i + m_i\theta_i \leq 1$ and so their product, one sees that the energy (69) is bounded by the mass of the original D$2l$-brane

$$E_{\text{total}} \leq M_{\text{D}2l}. \tag{70}$$

Moreover expanding eq (69) in θ_i series as

$$E_{\text{total}} = N + \sum_{i=1}^{l} \theta_i N_i + \sum \theta_{ij} N_{ij} + \cdots + \theta_{i_1\cdots i_l} N_{i_1\cdots i_l} \tag{71}$$

with

$$N = \prod_i n_i, \dots \theta_{i_1\cdots i_s} = \prod_{i=1}^{s} \theta_i; \quad N_i = \frac{m_i}{n_i} N; \quad N_{ij} = \frac{m_j}{n_j} N_i; \dots$$

$$N_{i_1 i_2 \cdots i_s} = \frac{m_{i_s}}{n_{i_s}} N_{i_1 i_2}\dots i_{s-1}, (1 \leq s \leq l-1); \quad N_{i_1\cdots i_l} = \prod_i m_i. \tag{72}$$

We will turn in a moment to this expansion, but now let us the tools we will need by giving the relations between the masses of D$2j$-branes $1 \leq j \leq l$. For a generic j, the masses of non-BPS branes on a non-commutative $\mathbb{T}_\theta^{2j}$ torii read as

$$M_{\text{D}2j} = \sqrt{2}\frac{\prod_{i=1}^{2j} R_i}{g_s\left(\alpha'\right)^{\frac{2j+1}{2}}} \left(\prod_{i=1}^{l} \left[1 + (2\pi\alpha' B_i)^2\right]^{\frac{1}{2}}\right); \tag{73}$$

while the mass of D0-branes is

$$M_{\text{D}0} = \sqrt{2}\frac{1}{g_s\left(\alpha'\right)^{\frac{1}{2}}}. \tag{74}$$

In eq (73) B_i denotes the $(2i-1, 2i)$ components of NS-NS $B_{\mu\nu}$ field which, in terms of θ_i, is given by

$$B_i = \frac{1}{2\pi R_{2i+1} R_{2i} \theta_i}. \tag{75}$$

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Note that the relations (73) may be derived from the relation $M_{\text{D}2j} = G_s C_{\text{D}2j}$ and the identity [9]

$$G_s = g_s \left(\frac{\det\left(g + 2\pi\alpha'B\right)}{\det B} \right)^{\frac{1}{2}} \tag{76}$$

where g_s is the closed string coupling constant.

Note also that in large non-commutativity, the $M_{\text{D}2j}$ masses can be linked with the mass $M_{\text{D}2l}$ of the original brane. Indeed, taking the large limits of all B_i's in eq (73) one gets

$$M_{\text{D}2l} = \sqrt{2} \frac{1}{\left(\prod_i^l \theta_i\right) g_s \left(\alpha'\right)^{\frac{1}{2}}} = \frac{1}{\left(\prod_i^l \theta_i\right)} M_{\text{D}0}. \tag{77}$$

If instead of taking all the B_i's large, we keep one of them, say B_n finite, one finds

$$M_{\text{D}2l} = \frac{1}{\left(\prod_{i\neq n}^l \theta_i\right)} M_{\text{D}2}^{(n)}. \tag{78}$$

with

$$M_{\text{D}2}^{(n)} = \sqrt{2} \frac{R_{2n-1}R_{2n}}{g_s \left(\alpha'\right)^{\frac{3}{2}}} \left[1 + \left(2\pi\alpha'B_n\right)^2\right]^{\frac{1}{2}}. \tag{79}$$

More generally taking $B_{i_1}, B_{i_2,}, ..., B_{i_s}$ large and $B_{i_{s+1}}, B_{i_{s+2},}, ..., B_{i_l}$ $(1 \leq s \leq l)$ finite and putting back in eq (73), one finds the following D-branes mass relations

$$M_{\text{D}2l} = \frac{1}{\left(\prod_{r=1}^s \theta_{i_r}\right)} M_{\text{D}2s}^{(n)}. \tag{80}$$

Substituting these relations into the developement (71), one gets an energy formula E_{total} expanded in terms of the masses $M_{\text{D}2j}$ of D2j-branes $(1 \leq j \leq l)$

$$E_{\text{total}} = NM_{\text{D}2l} + \sum_{i=1}^l N_i M_{\text{D}2}^{(i)} + \sum_{i,j=1}^l N_{ij} M_{\text{D}4}^{(ij)} + \ldots + N_{i_1\ldots i_l} M_{\text{D}0}. \tag{81}$$

In this stage one may ask what does this formula mean? As the energy is upper bounded by $M_{\text{D}2l}$ $(E_{\text{total}} \leq M_{\text{D}2l})$, it seems that the original unstable $M_{\text{D}2l}$ annihilates to different kinds of $M_{\text{D}2j}$ branes $(0 \leq j \leq l)$ in the vacuum. To interpret this spectrum; let us first consider the case of a non-BPS D2-brane on $\mathbb{T}_\theta^2$. In this case, which corresponds to $l = 1$ in eq (69), the energy eq(81) splits as follows

$$E\left(\mathcal{P}_{n+m\theta}\right) = nM_{\text{D}2} + mM_{\text{D}0}. \tag{82}$$

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Following [16], it is natural to interpret this spectrum as a bound state of n D2-branes and m D0-branes. This interpretation which is dictated by the study of tachyon condensation is also supported by T-duality and the analysis of the following exact mass spectrum of the $\{m\text{D0}, n\text{D2}\}$ bound state

$$M_{(n,m)} = \sqrt{2}\frac{R_1 R_2}{g_s \left(\alpha'\right)^{\frac{3}{2}}} \left[1 + \left(2\pi\alpha' B_{\text{eff}}\right)^2\right]^{\frac{1}{2}}, \tag{83}$$

where B_{eff} is an effective field, including the flux due to m D0-branes,

$$B_{\text{eff}} = B + \frac{1}{2\pi R_1 R_2}\frac{m}{n}. \tag{84}$$

Taking the large limit of $(2\pi\alpha' B_{\text{eff}})$, then the mass formula (82) is reproduced. Extending this study to the problem of non-BPS D2l-branes on the non-commutative torus $\mathbb{T}^{2l}_\theta$, our mass formula (81) may be understood as describing bound states of ND2l-branes, $\left(\sum_{i=1}^{l} N_i\right)D(2l-2)$-branes, $\left(\sum_{i,j=1}^{l} N_{ij}\right)D(2l-4)$-branes,$\cdots$, $\left(\sum_{i_1...i_s}^{l} N_{i_1...i_s}\right)D(2l-s)$-branes,$\cdots$, $\left(\sum_{i_1...i_s}^{l} N_{i_1...i_{l-1}}\right)$D2-branes and finally $\left(\prod_{i=1}^{l} m_i\right)$ D0-branes. Note that if one is considering the general situation where all the $2l$ radii R_{2i-1} and R_{2i} of the $2l$-dimensional torus are different, then one should distinguish various kinds of D2j-branes for a given j $(1 \leq j < l)$.

D2j branes in the bs	mass	# of D2j branes	Kind of D2j branes	Total # of D2j branes
D0	$\frac{\sqrt{2}}{g_s(\alpha')^{1/2}}$	$m_1 m_2$	1	$m_1 m_2$
D$^{(1)}$2	$\frac{\sqrt{2}R_1 R_2}{g_s(\alpha')^{3/2}}\Omega_1$	$n_1 m_2$	2	$n_1 m_2 + n_2 m_1$
D$^{(2)}$2	$\frac{\sqrt{2}R_3 R_4}{g_s(\alpha')^{3/2}}\Omega_2$	$n_2 m_1$		
D4	$\frac{\sqrt{2}R_1 R_2 R_3 R_4}{g_s(\alpha')^{5/2}}\Omega_1\Omega_2$	$n_1 n_2$	1	$n_1 n_2$

Table (b)

In table (b) we have given the complete spectrum of the non-BPS branes involved in the bound state (bs) $\{$D4,D2,D0$\}$. The general spectrum of $\{$D2l,D$(2l-2)$,...,D2,D0$\}$ bound state for all θ_i's irrational is given in table (c).

D2j branes in the bs	mass	# of D2j branes	Kind of D2j branes	Total # of D2j branes
D0	$\dfrac{\sqrt{2}}{g_s(\alpha')^{1/2}}$	$\mathrm{M}=\Pi_{i=1}^{l}m_i$	1	$\Pi_{i=1}^{l}m_i$
$\mathrm{D}^{(i)}2$	$\dfrac{\sqrt{2}R_{2i-1}R_{2i}}{g_s(\alpha')^{3/2}}\Omega_i$	$\dfrac{n_i}{m_i}M$	l	$n_1m_2+n_2m_1$
$\mathrm{D}^{(ij)}4$	$\dfrac{\sqrt{2}(R_{2i-1}R_{2i}\Omega_i)(R_{2j-1}R_{2j}\Omega_j)}{g_s(\alpha')^{5/2}}$	$\dfrac{n_in_j}{m_im_j}M$	$\dfrac{l(l-1)}{2}$	$M\Sigma_{ij}\dfrac{n_in_j}{m_im_j}$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
$\mathrm{D}^{\{i_1,\dots,i_s\}}2s$	$\dfrac{\sqrt{2}\Pi_{i=1}^{s}(R_{2i_s-1}R_{2i_s}\Omega_{i_s})}{g_s(\alpha')^{(2s+1)/2}}$	$M\Pi_{r=1}^{s}\left(\dfrac{n_{ir}}{m_{ir}}\right)$	$\dfrac{l!}{(l-s)!s!}$	$M\Sigma\{\Pi_{r=1}^{s}\left(\dfrac{n_{ir}}{m_{ir}}\right)\}$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
$\mathrm{D}2l$	$\dfrac{\sqrt{2}\Pi_{i=1}^{l}(R_{2i-1}R_{2i}\Omega_i)}{g_s(\alpha')^{(2l+1)/2}}$	$\Pi_{i=1}^{l}n_i$	1	$\Pi_{i=1}^{l}n_i$

Table (c)

Let us make two comments concerning this spectrum.

- Inspired from the D0-D2 bound state analysis, we can derive the exact mass formula $M_{\{n_1,\dots,n_l,m_1,\dots,m_l\}}$ of the $\{\mathrm{D}2l,\mathrm{D}(2l-2),\dots,\mathrm{D}2,\mathrm{D}0\}$ bound states. We show that

$$M_{\{\mathbf{n},\mathbf{m}\}} = \sqrt{2}\frac{\prod_{i=1}^{l}(R_{2i-1}R_{2i}n_i)}{g_s(\alpha')^{\frac{2l+1}{2}}}\left(\prod_{j=1}^{l}\left[1+(2\pi\alpha'B_{i\text{ eff}})^2\right]^{\frac{1}{2}}\right),\qquad(85)$$

  where $\mathbf{n}=(n_1,\dots,n_l)$, $\mathbf{m}=(m_1,\dots,m_l)$ and B_{eff} is an effective field given by

$$B_{i,\text{eff}} = B_i + \frac{1}{2\pi R_{2i-1}R_{2i}}\frac{m_i}{n_i};\quad i=1,\cdots,l.\qquad(86)$$

- In the case where some of the $\mathbb{T}^2_{\theta_i}$ factors of the non-commutative $2l$ dimensional torus are fuzzy torii, that is some of the θ_i's, $1\le i\le s$, are irrational and the remaining ones are rational; $\theta_i=q_i/p_i$ for $s+1\le i\le l$, one still has bound states of type $\{\mathrm{D}2s,\mathrm{D}2s-2,\dots,\mathrm{D}2,\mathrm{D}0\}$ whose spectrum mass may be read from table (c) in addition to extra D0-branes. In this case the tachyon soliton T corresponding to such representation is

$$T = t_{\max}.\left(\overset{s}{\underset{i=1}{\otimes}}\mathcal{P}_{\{n_i+m_i\theta_i\}}\right)\otimes\left(\overset{l}{\underset{i=s+1}{\otimes}}\Pi_{k_i}\right).\qquad(87)$$

  For level $\mathbf{k}_s=(k_{i_{s+1}},\dots,k_l)$ vacuum configurations on the fuzzy torus, the energy of the solitons is given by

$$E\left(\mathcal{P}_{\{n_i+m_i\theta_i\}},\Pi_{k_i}\right) = \left(\prod_{i=1}^{s}(n_i+m_i\theta_i)\right)\left(\prod_{i=s+1}^{l}\frac{k_i}{p_i}\right).\qquad(88)$$

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In what follows we want to show that the above configurations (87,88) do not have the same bound states and possed the following goup G as a full gauge symmetry

$$G = U \left(\prod_{i=1}^{s} k_i^{(ir)} \right) \times U \left(\prod_{i=s+1}^{l} k_i^{(r)} \right). \tag{89}$$

For bound states, this is clearly seen on the eq(88) since they appear from the Powers-Rieffel projectors in one to one correspondence with irrational θ_i's. For a fixed value of s, $(1 \le s \le l)$, one has bound states of type {D2s,D(2s−2),...,D2,D0} whose spectrum varies with s. Concerning the gauge symmetry, it is interesting to note first that for $s = 0$, $(h = 0)$, that is all θ_i's are rational. one is in presence of only D0-branes and so the gauge symmetry is $U \left(\prod_{i=1}^{l} k_i^{(r)} \right)$. For non-zero s, the situation is a little bit subtle since there exists bound states involving other kinds of D2j-branes for $0 \le j \le s$ in addition to the D0-branes of the rational factors having $U \left(\prod_{i=s+1}^{l} k_i^{(r)} \right)$ as a gauge group. The $U \left(\prod_{i=1}^{s} k_i^{(ir)} \right) \times \prod_{h=1}^{l} U \left(N_{i_1 i_2 ... i_h}^{(ir)} \right)$ gauge group of the D2j-branes for $0 \le j \le s$ system may be obtained from symmetry of the non-commutative 2l-dimensional torus. Using the T-duality transformations

$$\theta_i' = \frac{\alpha_i - \beta_i \theta_i}{\gamma_i - \delta_i \theta_i}; i = 1, \dots, l \tag{90}$$

$$\mathfrak{A}_{\theta_i'} \sim \mathfrak{A}_{\theta_i}, \quad \mathfrak{A}_{\theta'} \sim \mathfrak{A}_{\theta}, \tag{91}$$

with $\alpha_i \delta_i - \beta_i \gamma_i = 1$ and α_i, β_i, γ_i, $\delta_i \in \mathbb{Z}$; leaving the total energy (81) invariant. Under the above $[SL(2, \mathbb{Z})]^l$ dualities, it is usually possible to rewrite eqs (88) as

$$E(\mathcal{P}_{\{n_i+m_i\theta_i\}}, \Pi_{k_i}) = \left[\prod_{i=1}^{s} k_i^{(ir)} (\beta_i + \alpha_i \theta_i) \right] \left(\prod_{i=s+1}^{l} \frac{k_i^{(r)}}{p_i} \right); \tag{92}$$

where the k_i's $(1 \le i \le s)$ are the greatest commun divisor of (n_i, m_i) pairs. In the case of the D2-D0 bound state; $(n + m\theta) M_{D2}$ was interpreted in [16] as the energy of k bound states of D0-branes with a $U(k)$ gauge symmetry. Extending this reasoning to the non-commutative torus $\mathbb{T}_\theta^{2l}$ and using eq (92), the total number of D0-branes is $\prod_{i=1}^{s} \left(k_i^{(ir)} \right)$ coming from the irrational case and $\prod_{i=s+1}^{l} \left(k_i^{(r)} \right)$ from the rational one. Seen as two different sets of D0-branes, one concludes that the gauge symmetry is $\prod_{i=1}^{s} \left(k_i^{(ir)} \right) \times \prod_{i=s+1}^{l} \left(k_i^{(r)} \right)$. If we assume that all D0-branes of the soliton are indistinguishable, then we end with a larger gauge group $\prod_{i=1}^{l} (k_i)$ containing the previous symmetries as subgroups.

5   Discussion and Conclusion

In this paper we have studied the solitonic solutions of a non-BPS D2l-branes; $l \ge 1$; on both Moyal spaces R_θ^{2l} and non-commutative torii T_θ^{2l}. Actually this study may

also be viewed as an extension of the analysis of the tachyon condensation made in [16] for non-BPS D2-branes in presence of a constant NS-NS B-field. Our results may be summarised into the following two:

(1) We have derived soliton solutions for non-BPS D2l-branes; $l \geq 1$; on R_θ^{2l}. In particular, we have shown that besides the usual stable vacuum field configuration, there exists also quasi-stable solutions minimising the total energy $E_{\text{total}} = E(\theta_1, \theta_2, ..., \theta_l)$. These solutions are associated with local minima of E_{total}. and are interpreted as non-BPS D2s_1-branes, $0 \leq s_1 \leq l-1$, representing the l decay levels of the original non-BPS D2l-brane into lower dimensional world volume branes. Besides the original non-BPS D2l-brane and the non-BPS D0-branes one ends up with $2^l - 2$ brane configurations partionned as

$$
\begin{aligned}
[D2l] \; \curvearrowright \; & \sum_{s_1=1}^{l-1} \frac{l!}{(l-s_1)!s_1!}[D2s_1] \\
= \; & l[D2] + \frac{l(l-1)}{2}[D4] + \frac{l(l-1)(l-2)}{6}[D6] + ... \\
& + \frac{l(l-1)(l-2)}{6}[D(2l-6)] + \frac{l(l-1)}{2}[D(2l-4)] + l[D(2l-2)]. \quad (93)
\end{aligned}
$$

This expansion means that at a given stage of the condensation, say at a step j, $(0 < j < l)$, one has $\frac{l!}{(l-j)!j!}$ kinds of D2j-branes. For $j = 1$ and $j = l-1$ for instance, we have respectively l world volumes D2-branes and D$(2l-2)$-branes and for $j = 2$ and $j = l-2$ we have $\frac{l(l-1)}{2}$ world volumes $D4$-branes and a similar number of $D(2l-4)$-branes. In figure 1 we have proposed a shape of the $V(\phi)$ potential with l waves to describe these quasi-stable solutions.

Moreover, since for a generic extremum s_1, the corresponding D2s_1-brane, $1 \leq s_1 \leq l-1$ is an unstable configuration, one can imagine that D2s_1 itself condensates into lower dimensional world volumes D2s_2-branes, $0 \leq s_2 \leq s_1 - 1$. Thus, given an integer s_2, we have here also

$$
[D2s_1]- \; \curvearrowright \; \sum_{s_2=1}^{s_1-1} \frac{s_1!}{(s_1 - s_2)!s_2!}[D2s_2].
$$

More generally we have the following result: Starting from the original non-BPS D2l-brane on R_θ^{2l}; with $l \geq 1$; and taking the weaker definition of large non commutativity which ammounts to put one of the θ_i' s large and all remaining others finite, one gets l kinds of $(2l-2)$-dimensional solitons identified with D$(2l-2)$-branes. Repeating the same mechanism to each one of the l D$(2l-2)$-branes, one obtains, after integrating out equivalent configurations, $\frac{l(l-1)}{2}$ D$(2l-4)$-branes on the $R_\theta^{2(l-2)}$ Moyal space. Successive iterations lead at the end to $\Pi_{s=0}^l (\frac{l!}{(l-s)!s!})$ D0-branes. Furthermore if we denote by k_s, the level of the D2s solitons with $k_s \geq 1$ and $k_l = 1$ one gets $U(\Pi_{s=0}^l [\frac{k_s l!}{(l-s)!s!}])$ as a full gauge symmetry.

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(2) We have studied the tachyon solitons on $2l$-dimensional non-commutative torii using both rational and irrational representations. We have determined the solitons mass spectrum and shown for s irrational θ_i's, $1 \leq s \leq l$, the existence of general bound states $\{D2s, D(2s-2),...,D2i,..., D2, D0\}$, extending the D2-D0 bound state obtained in [16] for the case of a non-BPS D2-brane on an irrational two torus. From the mass formula of our bound state, namely

$$M_{\{\mathbf{n}_s,\mathbf{m}_s\}} = \sqrt{2}\frac{\prod_{i=1}^{s}(R_{2i-1}R_{2i}n_i)}{g_s(\alpha')^{\frac{2l+1}{2}}}\left(\prod_{j=1}^{s}\left[1+(2\pi\alpha'B_{i\text{ eff}})^2\right]^{\frac{1}{2}}\right), \qquad (94)$$

with $\mathbf{n}_s = (n_1, n_2..., n_s)$ and $\mathbf{m}_s = (m_1, m_2..., m_s)$, or equivalently by using eq(83)

$$M_{\{\mathbf{n},\mathbf{m}\}} = \prod_{i=1}^{s}\{M_{(n_i,m_i)}\}, \qquad (95)$$

one learns that $\{D2s, D(2s-2), ..., D2i, ..., D2, D0\}$ system is unstable and decays into s $\{D2-D0\}^{(i)}$ bound states of masses $M_{(n_i,m_i)}$. Note that for the case $s = 0$; i.e. no θ_i is irrational, we have $D0$-branes but no bound state as expected. For $s = l$; i.e. all θ_i s are irrational, we have bound states type $\{D2l, D(2l-2), ..., D2s, ..., D2, D0\}$ which as suggested desintegrate into l $\{D2-D0\}$ bound states. Applying the Bars *et al* analysis made for the $D2-D0$ bound state to the $\{D2s, D(2s-2), ..., D2i, ..., D2, D0\}$ system, it is no difficult to check that the full gauge group of the vacuum configuration is $U(\Pi_{i=1}^{s}k_i^{(ir)}\Pi_{i=s+1}^{l}k_i^{(r)})$. It contains as a subgroup $U(\Pi_{i=1}^{s}k_i^{(ir)}) \times U(\Pi_{i=s+1}^{l}k_i^{(r)})$, the gauge symmetry coming from the irrational and rational sectors. The potential describing solitons on T_θ^{2l} has a similar shape as in figure 1 except that one has now to specify:

- The representation considered for T_θ^{2l}: rational or irrational.

- The interpretation of the extrema in terms of bound states. Extension of these results to other cases such as non-commutative orbifods will considered in a future occasion.

Acknowlegement

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